No Negative Modes About the Axionic Wormhole Instanton

Alex Kinsella

(Advisor: Sean Hartnoll, Second Reader: Shamit Kachru)

(Dated: June 5, 2015)

Abstract

The axion is a hypothetical elementary particle originally proposed as a solution to the strong CP problem. Because the axion is associated with a pseudo-scalar field instead of a scalar field, it has a nontrivial instanton solution, which has the form of a spherically symmetric wormhole. We examine the perturbations about this instanton using a scalar-vector-tensor decomposition. The vector modes are pure gauge, and therefore may be directly integrated. We compute the action for the tensor modes and the spectrum is shown to be non-negative. We obtain the action of the scalar modes using a transformation from the known action of a scalar field coupled to gravity. The scalar mode spectrum is also shown to be non-negative, so the axionic wormhole instanton has no negative modes. This discourages the interpretation of this instanton as a decay rate, and instead supports the Coleman interpretation of this instanton as showing that the ground state is a superposition of many-universe states.

I. INTRODUCTION

One of the most important currently unsolved problems in physics is the strong CP (charge-parity) problem. Quantum chromodynamics allows for violation of CP symmetry, but no such violation is observed in nature, giving rise to a fine-tuning problem - if a violation is allowed but not observed, then some parameter must be "tuned" to exactly or very close to zero. The best way to explain such a fine-tuning is to show that the parameter was not free after all, and that it is forced by some mechanism to take an exact value. The leading solution to the CP problem was proposed by Peccei and Quinn in 1977, and consists of the introduction of a new scalar particle called the *axion* [1]. Peccei and Quinn proposed a new symmetry which would be spontaneously broken, such that it has the axion as its pseudo Goldstone boson. (The axion must be a pseudo Goldstone boson because it acquires a small mass from instantons, but a true Goldstone boson must be massless.) Because the axion has a small mass and interacts only weakly with other matter, it has also become a candidate component of dark matter [2].

The existence of an axion field has potentially strange consequences for cosmology. When coupled to gravity, axions give rise to solutions to Einstein's equation that suggest the existence of either many-universe states or instability of the universe. This prompted worry from theorists that, through decay of the universe, the axion may lead to non-conservation of probability, baryon number, and other quantities that ought to be conserved [3]. In this paper, we investigate the spectrum of one-loop fluctuations about the background instanton and show that there is no negative mode, meaning that the instanton does not represent a decay rate, and the worries are unfounded. Instead, these solutions represent the fact that the ground state is a superposition of many-universe states.

The layout of the rest of this paper is as follows. Section 2 is a review of Euclidean methods for semiclassical quantum gravity, which we will use to study the axion field. Section 3 introduces the wormhole solution obtained for an axion field coupled to gravity, and the properties of this solution will be the focus of the rest of the paper. Section 4 introduces the scalar-vector-tensor decomposition of perturbations about the solution and discusses the vector perturbations. Sections 5 and 6 cover the tensor and scalar modes, respectively. Finally, section 7 contains discussion and consequences of the results and section 8 is a conclusion.

II. SEMI-CLASSICAL QUANTUM MECHANICS AND GRAVITY

We begin with a summary of semi-classical quantum gravity, the framework we will use to examine the gravitational effects of the instanton. Semi-classical quantum gravity is obtained by naively quantizing general relativity and then applying a semi-classical approximation. Because of this, its formalism is very similar to that of semi-classical quantum mechanics, so we will first develop the latter theory.

Semi-classical quantum mechanics is naturally expressed in the *path integral formulation* of quantum mechanics. The path integral formulation is a third way to present quantum mechanics, alongside wave mechanics and matrix mechanics, in which systems essentially take all paths in state space from one state to another (see fig. 1). To explain the formalism, we will restrict our attention to a single particle moving in one dimension. The path integral formulation postulates that the transition amplitude for the particle to travel from position a at time t_i to position b at time t_f is given by a sum over all possible paths $x(t)$ such that $x(t_i) = a$ and $x(t_f) = b$. Each path is to be added with equal magnitude, but with a phase proportional to the classical action of the path, $S[x(t)]$. This sum is represented mathematically by a "path integral" as follows:

$$K(a, b; t_i, t_f) = N \int_{x(t_i)=a}^{x(t_f)=b} \mathcal{D}x \; e^{\frac{i}{\hbar} S[x(t)]} \tag{1}$$

where N is a normalization constant that will not be important for our purposes. In the limit that $\hbar \to 0$, we may apply the stationary phase approximation to find that the dominant contributions to this integral will be paths for which the classical action is stationary. This gives rise to the equations of motion $\delta S = 0$ which, for a particle in space, are the Euler-Lagrange equations. So we see that in the limit $\hbar \to 0$, we recover the classical behavior that we expect.

Now we will investigate the semi-classical limit. For $\hbar$ small but nonzero, we consider the leading order corrections to the probability amplitude, which come from paths that differ only infinitesimally from the classical path. Now that we have made our semi-classical approximation, we will work in units where $\hbar = 1$ to help declutter our equations. We treat these almost-classical paths perturbatively, writing $x(t) = x_{cl}(t) + \delta x(t)$, where $\delta x(t)$ is infinitesimal and is equal to zero at the endpoints. We may expand the action in a series

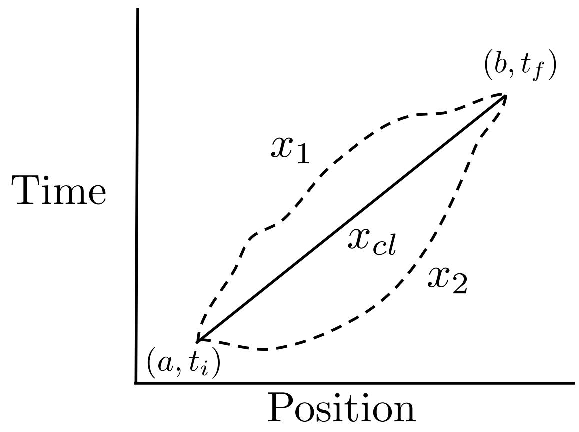

Figure 1. Schematic representation of the path integral. The path x_{cl} is the classical path between point a at time t_i and point b at time t_f. The paths x_1 and x_2 are also paths between these points, which contribute to the probability amplitude. If these paths are infinitesimal perturbations of the classical path, they will contribute to the semi-classical amplitude.

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as:

$$S[x(t)] = S[x_{cl}(t)] + \frac{\delta S}{\delta x}[x_{cl}(t)]\delta x(t) + \frac{1}{2}\frac{\delta^2 S}{\delta x^2}[x_{cl}(t)]\left(\delta x(t)\right)^2 + O\left((\delta x(t))^3\right) \tag{2}$$

We know that $\delta S = 0$ for a classical path, so the first order term vanishes, and we are left with the second order term as the leading correction. We call $\frac{\delta^2 S}{\delta x^2}$ the *quadratic action*. Our path integral becomes, assuming only one classical path:

$$K(a, b; t_i, t_f) \approx N \int_{x(t_i)=a}^{x(t_f)=b} \mathcal{D}(\delta x)\ e^{i\left(S[x_{cl}(t)] + \int \mathrm{d}t\ \frac{1}{2}\frac{\delta^2 S}{\delta x^2}[x_{cl}(t)](\delta x(t))^2\right)} \tag{3}$$

For multiple classical paths, the amplitude is simply a sum over all of them.

Now that we are equipped with a semi-classical approximation, we want to investigate tunneling phenomena. To do so, we must investigate solutions to the equation of motion that are not considered in the other formulations of quantum mechanics [4]. In the same way that integrals along the real line may be dominated by a saddle point elsewhere in the complex plane, complex solutions to the equations of motion $\delta S = 0$ may provide dominant contributions to the path integral. Information about tunneling is contained in the semi-classical approximations about the complex solutions to $\delta S = 0$ that have finite action (meaning that they do not tunnel through a thick infinite barrier) and are real functions of *imaginary time,* $\tau \equiv it$. We call these solutions *instantons* and the transformation $t \to -i\tau$ is called a *Wick rotation.*

The classical action of our particle is given by:

$$S = \int \mathrm{d}t\mathrm{d}^3x \; \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - V(\phi) \right] \tag{4}$$

When we perform a Wick rotation, we change variables in the action to obtain:

$$\begin{aligned} S &= \int -i\mathrm{d}\tau\mathrm{d}^3x \; \left[\frac{1}{2} \left(\frac{\partial \phi}{-i\partial \tau} \right)^2 - V(\phi) \right] \\ &= i \int \mathrm{d}\tau\mathrm{d}^3x \; \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + V(\phi) \right] \\ &\equiv iS_E \end{aligned}$$

where in the last line we have defined the *Euclidean action.* The word "Euclidean" refers to the fact that Wick rotation replaces the time-like coordinate t with the space-like coordinate τ, so that spacetime becomes Euclidean. The rotation made two important changes to the action: the first is that it brought a factor of i out front, which will give a possibility of convergence in the functional integral. The other change is the relative sign between the kinetic and potential terms. This change makes it so that a particle in imaginary time essentially sees the potential upside-down, giving rise to new classical solutions, the instantons.

Let's look at the probability amplitude in imaginary time:

$$\begin{aligned} K_E(a, b; \tau_i, \tau_f) &= N \int_{x(\tau_i)=a}^{x(\tau_f)=b} \mathcal{D}x \; e^{iS[x(\tau)]} \\ &= N \int_{x(\tau_i)=a}^{x(\tau_f)=b} \mathcal{D}x \; e^{-S_E[x(\tau)]} \end{aligned}$$

As promised, the integrand is now a decaying exponential instead of an oscillatory one, so some integrals may now converge. In particular, let's look at our semi-classical approximation about an instanton $x_{cl}(\tau)$:

$$K_E(a, b; \tau_i, \tau_f) \approx N \int_{x(\tau_i)=a}^{x(\tau_f)=b} \mathcal{D}(\delta x) \; e^{-\left(S_E[x_{cl}(\tau)] + \int \mathrm{d}\tau \; \frac{1}{2} \frac{\delta^2 S}{\delta x^2} [x_{cl}(\tau)] (\delta x(\tau))^2 \right)} \tag{5}$$

The second term in the exponential is Gaussian, and we may, in fact, perform a Gaussian integral here [4]. The result is:

$$K_E(a, b; \tau_i, \tau_f) \approx N \frac{1}{\sqrt{\det'\left(\frac{\delta^2 S}{\delta x^2}\right)}} e^{-S_E[x_{cl}(\tau)]} \tag{6}$$

where we write the determinant with a prime to denote the product of non-zero eigenvalues of the operator $\frac{\delta^2 S}{\delta x^2}$. Again, for multiple instantons, we sum this formula over all of them.

Now we will see how this tells us about semi-classical phenomena. We will switch to bra-ket notation, in which

$$K_E(a, b; \tau_i, \tau_f) = \left\langle a \mid e^{-H(\tau_f - \tau_i)} \mid b \right\rangle \tag{7}$$

We may expand the right hand side in a complete set of eigenstates to find:

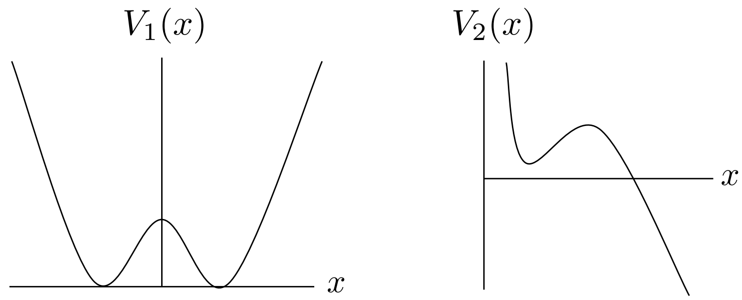
$$\left\langle a \mid e^{-H(\tau_f - \tau_i)} \mid b \right\rangle = \sum_n e^{-E_n(\tau_f - \tau_i)} \langle a \mid n \rangle \langle n \mid b \rangle \tag{8}$$

Now if we take $(\tau_f - \tau_i)$ very large (i.e. we look at large imaginary times) the lowest energy eigenstate dominates the sum, and we are able to obtain the ground state energy and wave function as the leading term in the sum. In particular, we will see the effect of tunneling on the ground state energy. Because we did not change the Hamiltonian, the Euclidean system has the same energy eigenvalues as the system in real time, so we may use instantons to learn about semi-classical effects of the real-time system. Recall that quantum tunneling lifts the degeneracy of classical ground states by creating a ground state that is the symmetric combination of the particle located in each ground state, while the first excited state is the antisymmetric combination. The instanton gives the energy splitting between these two lowest states as:

$$\Delta E \sim \frac{1}{\sqrt{\det'\left(\frac{\delta^2 S}{\delta x^2}\right)}} e^{-S[x_{cl}(\tau)]} \tag{9}$$

where we have omitted a constant normalization factor. See [4] for details of this calculation. An example of a system in which this calculation may be applied is the double well potential (see fig. 2).

Consider what happens if the quadratic action has exactly one negative eigenvalue. In this case, the square root prefactor will be pure imaginary, and the ground state energy will be complex. An imaginary part in the energy is interpreted as a decay rate, because of the time evolution $e^{iE_n t}$. This is the case of metastable decay, where a quantum particle tunnels



$$\Delta E \sim \frac{1}{\sqrt{\det'\left(\frac{\delta^2 S}{\delta x^2}\right)}} e^{-S[x_{cl}(\tau)]} \qquad \Gamma \sim \frac{1}{\left|\sqrt{\det'\left(\frac{\delta^2 S}{\delta x^2}\right)}\right|} e^{-S[x_{cl}(\tau)]}$$

Figure 2. Two examples of potentials in which a nontrivial instanton exists for the quantum system. In Euclidean time, the particle sees the potential upside-down, so new classical solutions exist. For V_1, the double well, the instanton begins at the bottom of the left well and rolls "down and up" the middle hill, ending at the right well. The action of this instanton, along with the determinant of its quadratic action, give the energy splitting of the two lowest states. The potential V_2 is metastable, and the instanton represents decay via tunneling. The instanton begins at the bottom of the metastable well and rolls "down" to the right, stops at the classical turning point, and returns to the well. Because of this out-and-back behavior, this instanton is called a *bounce*.

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out of a finite potential well (see fig. 2). In this case, we have that the decay rate Γ is given by:

$$\Gamma \sim \frac{1}{\left|\sqrt{\det'\left(\frac{\delta^2 S}{\delta x^2}\right)}\right|} e^{-S[x_{cl}(\tau)]} \tag{10}$$

Again, see [4] for details. There is always exactly one negative eigenvalue of the quadratic action for instantons that represent metastable decay, and no negative eigenvalues for instantons that represent tunneling between stable minima.

Now we transfer all of this formalism to semi-classical quantum gravity. We may extend the path integral to a general quantum system by generalizing paths in physical space to paths in the state space of the system, and by using the action appropriate to the system.

To quantize a classical system, we simply use the classical action. This allows us to write down a naive quantum theory of gravity that ultimately proves to be incorrect, but can be used to investigate semi-classical effects. We take as our action the Einstein-Hilbert action of general relativity along with the Lagrangian of any matter fields at play:

$$S = \frac{1}{16\pi G}\int_M \mathrm{d}t\mathrm{d}x\,\sqrt{-g}\,(-R+\mathcal{L}_{\text{matter}}) \tag{11}$$

where G is Newton's constant, M is the space-time manifold under consideration, g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, and $\mathcal{L}_{\text{matter}}$ is the Lagrangian of the matter fields. For variations of the metric that vanish and whose normal derivatives vanish on the boundary of M, this action is stationary if and only if the Einstein equations are satisfied. (In this work, we will not worry about boundary terms in the action or elsewhere. Boundary terms serve only to maintain the boundary conditions, which we will instead enforce by hand. They are also important when finding the value of the on-shell action $S[x_{cl}]$, but we are not interested in that quantity here.) Our path integral is given by:

$$Z = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\exp\left(iS[g_{\mu\nu},\phi]\right) \tag{12}$$

There are multiple problems with this path integral. First, it is not convergent. This is not enough to compromise its use, because the convergence of any path integral is dubious. However, for all path integrals that we believe represent the real world, it is possible to tame the divergence by introducing a finite number of parameters, which is called *renormalization.* However, if we expand the above integrand in a Taylor series, each order introduces more divergent terms. The divergences cannot be tamed by introducing a finite number of parameters, meaning that our candidate theory of quantum gravity is non-renormalizable. This makes it unacceptable as a quantum field theory describing physical reality. However, we may trust the path integral up to second order in the action, so that we may use it for a semi-classical approximation, which is valid for scales much larger than the Planck length, and for curvatures with associated radii much larger than the Planck length.

The instanton formalism carries through in its entirety from quantum mechanics to quantum gravity, but now an instanton is a solution to the Einstein equations, i.e. a metric for a universe, called a *gravitational instanton.* A gravitational instanton must have finite action, like an instanton in quantum mechanics, and it must be asymptotically Euclidean, meaning that it must approach flat space in the far past and far future. A negative mode about a gravitational instanton represents an instability of the universe itself. The most well-known gravitational instanton is the *Coleman-de Luccia instanton*, which has a negative mode describing the instability of a metastable false vacuum [5].

III. THE EUCLIDEAN WORMHOLE SOLUTION

Now we use the semi-classical techniques developed above to investigate the gravitational effects of an axion field. The axion field is naturally expressed using a three-form field strength $H_{\mu\nu\lambda}$. However, we will work with the local potential ϕ instead, defined by $*H = \mathrm{d}\phi$, where $*$ is the Hodge-dual. This makes ϕ a pseudo-scalar field. The Euclidean action for an axion field ϕ coupled to gravity is given by:

$$S = \frac{1}{2\kappa}\int \mathrm{d}\tau \mathrm{d}^3x\ \sqrt{g}\left(-R - (\nabla_\mu\phi)(\nabla^\mu\phi)\right) \tag{13}$$

where $\kappa = 8\pi G$. The sign of the kinetic term is opposite that of a standard scalar field in Lorentzian metric. This is because ϕ is a pseudo-scalar field instead of a scalar field, and psuedo-scalar fields change sign under Wick rotation [6].

Given this action, we search for an instanton by looking for a solution of the classical equations of motion that is asymptotically Euclidean. To simplify our derivation, we will use the spherically symmetric ansatz:

$$\mathrm{d}s^2 = a^2(t)\left(\mathrm{d}t^2 + \mathrm{d}\Omega_3^2\right) \tag{14}$$

where $\mathrm{d}\Omega_3^2$ is the metric on a 3-sphere. We vary our action with respect to g and ϕ to obtain the Einstein equations and the scalar field equation of motion, into which we plug our ansatz. The Einstein equations become:

$$\begin{aligned}\ddot{a} - a\left(1 + \frac{1}{12}\dot{\phi}^2\right) &= 0\\ \dot{a}^2 - a^2\left(1 + \frac{1}{12}\dot{\phi}^2\right) &= 0\end{aligned}$$

and the scalar field equation is:

$$\ddot{\phi}^2 + \dot{\phi}^2\left(4 - \frac{1}{3}\dot{\phi}^2\right) = 0 \tag{15}$$

where the dot denotes derivatives with respect to τ. We may eliminate $\dot{\phi}$ in the Einstein equations to obtain a differential equation for a, which reads:

$$\ddot{a}a + \dot{a}^2 - 2a^2 = 0 \tag{16}$$

We will solve this equation along with asymptotically flat boundary conditions, which say that as $\tau \to \pm\infty$, the metric approaches that of flat space: $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_3^2$. The general solution to our ODE is:

$$a(\tau) = \sqrt{c_1 e^{2\tau} + c_2 e^{-2\tau}} \tag{17}$$

As $\tau \to \infty$, this approaches the form $a(\tau) \to \sqrt{c_1} e^{\tau}$, and as $\tau \to -\infty$ this approaches the form $a(\tau) \to \sqrt{c_2} e^{-\tau}$. We can see that these both represent flat space via the coordinate transformations $\tau = \pm \log \rho$. We therefore have both constants free. However, one of them is redundant, because we may rescale our τ coordinate such that $c_1 = c_2$. Define $r \equiv \sqrt{c_1}$. We then find:

$$a(\tau) = r\sqrt{2\cosh 2\tau} \tag{18}$$

so that our metric takes the form:

$$\mathrm{d}s_{cl}^2 = 2r^2 \cosh(2\tau) \left(\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\right) \tag{19}$$

This instanton looks like a wormhole (see fig. 3). It connects two asymptotically flat universes, which would be classically degenerate ground states for this system. The reader should keep in mind that this is a wormhole in imaginary time, and therefore not a physical solution, so a question such as "could one traverse this wormhole?" does not make sense.

Because there are two fields, g_{uv} and ϕ in our action, an instanton consists of a configuration of both fields. Our equations of motion allow us to find the instanton value of the axion field from our knowledge of $a(\tau)$. We solve for $\dot{\phi}^2$ in our first Einstein equation to find:

$$\begin{aligned} \dot{\phi}^2 &= 12\left(\frac{\ddot{a}}{a} - 1\right) \\ &= 12\mathrm{sech}^2(2\tau) \end{aligned}$$

which we may integrate to find:

$$\phi_{cl}(\tau) = \sqrt{12}\,\arctan\tanh\tau \tag{20}$$

This function is plotted in fig. 4.

The most important question in the interpretation of this instanton is whether or not it has exactly one negative mode. This instanton represents tunneling between disconnected universes, but it is not clear from the solution itself whether the universes are stable or metastable states. If the instanton has exactly one negative mode, then one of the states is

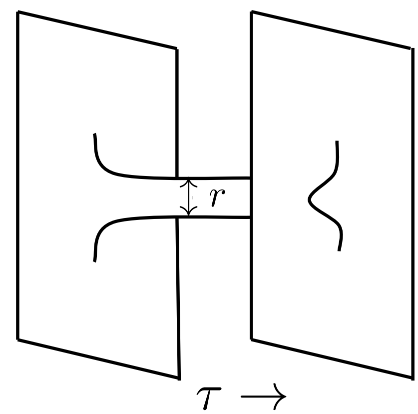

Figure 3. A schematic representation of the instanton metric, which resembles a wormhole. Two dimensions are suppressed in this picture; the constant τ cross sections are actually 3-spheres. The wormhole connects two asymptotically flat regions, which classically would be disconnected degenerate ground states. The minimal radius of the throat of the wormhole is a free parameter r.

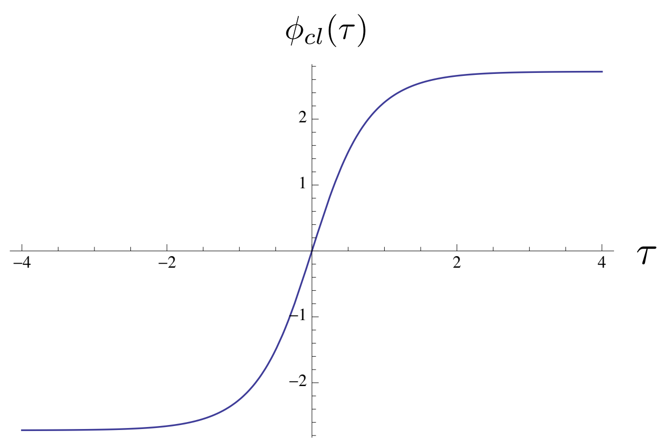

Figure 4. A plot of the instanton configuration of the axion field. This field configuration accompanies the wormhole configuration of the instanton metric. As is typical for an instanton, the axion field asymptotes to constant values for $\tau \rightarrow \pm\infty$ and undergoes a sharp transition in between (this is what gives rise to the name "instanton").

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actually metastable, and the universe would decay with a rate Γ given by the action of the instanton. We will now investigate this question using perturbation theory and show that there is no negative mode, so that the tunneling is between stable states and there is no decay.

IV. DECOMPOSITION OF PERTURBATIONS

To understand the interpretation of our instanton, we need to examine the spectrum of its quadratic action. To derive the quadratic action, we consider perturbations of our action, i.e. the effect on the action of the transformations $g \to g + \delta g$ and $\phi \to \phi + \delta\phi$, where δg and $\delta\phi$ are small. The metric perturbation δg can be an arbitrary tensor, but we will simplify our work by decomposing δg into scalar, vector, and tensor components. Using the representation theory of $SO(3)$, any tensor can be split into scalar, vector, and tensor modes that do not mix to linear order (in our case, to first order in $\hbar$) [7]. This is because the tensor representation of $SO(3)$ is reducible, and may be decomposed as a direct sum of irreducible representations, which may be thought of as scalars, transverse vectors, and trace-free, transverse, symmetric tensors.

We begin by splitting our perturbation into scalar, vector, and tensor parts, so that we may write our perturbed line element as:

$$\delta g = a^2(t)\left((1+2A)\mathrm{d}\tau^2 + S_i \mathrm{d}x^i \mathrm{d}\tau + \left[\gamma_{ij} + h_{ij}\right]\mathrm{d}x^i \mathrm{d}x^j\right) \tag{21}$$

where A, B, and ψ are scalar fields, S_i is a vector field, h_{ij} is a tensor field, and i, j run over spatial indices. We must further decompose our vector field into its divergence (a scalar) and a transverse vector, i.e. a vector that satisfies $\nabla_i V^i = 0$ (where the divergence is with respect to the spatial metric). We then have:

$$S_i = \nabla_i B + V_i \tag{22}$$

where B is a scalar and V_i is a transverse vector. Additionally, we decompose our tensor field into tensor field into a trace part, a scalar part, a transverse vector part, and a transverse, trace-free, symmetric tensor part. This looks like:

$$h_{ij} = -2\psi\gamma_{ij} + 2\nabla_i\nabla_j E + 2\nabla_{(i}F_{j)} + t_{ij} \tag{23}$$

where ψ and E are scalars, F_i is a transverse vector, t_{ij} is a transverse, trace-free symmetric tensor, and subscripts in parentheses denotes symmetrization. We have now decomposed our tensor metric perturbation into four scalars, two transverse vectors, and one transverse, trace-free tensor. A 4×4 symmetric tensor has ten degrees of freedom, so we should ensure that we still have as many. Each scalar contributes one degree of freedom. Each transverse vector is three degrees of freedom with one constraint, so two degrees of freedom total. A symmetric 3×3 tensor has six degrees of freedom, and transverse and trace-free conditions give four constraints, so this has two net degrees of freedom. In total we have: $4+2\cdot 2+2 = 10$ degrees of freedom, just like we ought to.

Because the scalar, vector, and tensor parts of the perturbation do not mix to first order, we may investigate the spectrum of the quadratic action by separately inspecting each of these three parts. The simplest is the vector part. Like electromagnetism, gravity has gauge freedom. But while electromagnetism has just one gauge degree of freedom, corresponding to a phase angle, gravity has four gauge degrees of freedom, corresponding to the choice of four functions that determine the coordinates on the four dimensional manifold of spacetime. In our problem, it turns out that two of these redundant degrees of freedom are in the vector part of the perturbations. This has the effect of making V_i a Lagrange multiplier, meaning that V_i only appears linearly in the quadratic action. This allows us to integrate over V_i, giving a delta functional that serves to enforce the momentum constraints, which are components of Einstein's equation that provide constraints on the perturbations. We may then use the delta functional to perform the F_i integral as well (after introducing the canonical momentum conjugate to F_i), and end up with a number called the *gauge orbit volume*. See appendix B of [8] for details of this calculation.

Because the vector path integral could be explicitly computed, the vector perturbations do not contribute any eigenvalues. In particular they do not contribute any negative eigenvalues. We now move on to the tensor and scalar modes.

V. TENSOR MODES

We will next consider the tensor modes, for which the metric perturbation is given by:

$$\delta g_T = t_{ij} \mathrm{d}x_i \mathrm{d}x_j \tag{24}$$

where t_{ij} is a transverse, traceless tensor. To derive the second order action resulting from this perturbation, we plug the above form into our equation for the action and expand the result to second order. This calculation is done in [8] for a scalar field coupled to gravity. Because the axion field perturbation is a scalar mode, it does not enter into the quadratic action for tensors, so the tensor quadratic action is identical for a scalar field and for an axion field. We therefore write down the quadratic action for a scalar field obtained in [9]:

$$S_{2,T} = \frac{1}{8\kappa}\int \mathrm{d}\tau \mathrm{d}^3x\ \sqrt{\gamma}a^2\left[h'^{ij}h'_{ij} + \nabla^i h^{jk}\nabla_i h_{jk} + 2h^{ij}h_{ij}\right] \tag{25}$$

where $a(\tau)$ is the conformal factor from the instanton metric, γ is the determinant of the background metric on the 3-sphere, and covariant derivatives and contractions are with respect to this metric.

We want to diagonalize this action to search for negative modes. To do so, we first rescale our perturbations using $\tilde{h}_{ij} = ah_{ij}$, which gives us

$$\begin{aligned} \nabla_i h^{ij} &= \nabla_i \tilde{h}^{ij} \\ ah'^{ij} &= \frac{\tilde{h}'^{ij}a - a'\tilde{h}^{ij}}{a} \end{aligned}$$

We plug these into our action and integrate by parts to find:

$$S_{2,T} = \frac{1}{8\kappa}\int \mathrm{d}\tau \mathrm{d}^3x\ \sqrt{\gamma}\tilde{h}^{ij}\left(-\frac{d}{dt^2} + \frac{a''}{a} - \Delta_3 + 2\right) \tag{26}$$

We write this as:

$$S_{2,T} = \frac{1}{8\kappa}\int \mathrm{d}\tau \mathrm{d}^3x\ \sqrt{\gamma}\tilde{t}^{ij}\left[\hat{U} - \Delta_3 + 3\right]\tilde{t}_{ij} \tag{27}$$

where

$$\hat{U} = -\frac{d^2}{d\tau^2} + \frac{a''}{a} - 1 = -\frac{d^2}{d\tau^2} + \mathrm{sech}^2(2\tau) \tag{28}$$

We recognize this as a Schrodinger operator. Because $\hat{U}$ acts only on τ and $-\Delta_3$ acts only on spatial variables, the two operators commute, so we may find their spectra separately and add them. To search for eigenfunctions of $\hat{U}$, we need boundary conditions. We may use our knowledge of Schrodinger operators to note that an eigenfunction of a negative eigenvalue will, at infinity, take values less than that of the potential, and will therefore be a real exponential. We want the well-behaved solution, which is exponential decay. So we see that our boundary condition is that the eigenfunction approaches zero as $\tau \to \pm\infty$.

The "potential function" $\text{sech}^2(2\tau)$ is strictly positive, and therefore $\hat{U}$ has strictly positive eigenvalues for solutions that approach zero as $\tau \to \pm\infty$, as is well known from quantum mechanics. The tensor Laplacian $-\Delta_3$ has eigenvalues $l(l+2)-2$, for integers $l \geq 2$ [10]. In particular, the smallest eigenvalue is 6. We therefore see that the tensor quadratic action has no negative modes. Now we move on to the scalar modes.

VI. SCALAR MODES

The scalar mode perturbations are given by:

$$\delta g_S = \left(-(1+2A)\mathrm{d}t^2 + \nabla_i B \, \mathrm{d}x^i \mathrm{d}t + (\psi\gamma_{ij} + \nabla_i\nabla_j E)\, \mathrm{d}x^i \mathrm{d}x^j\right) \tag{29}$$

The second order action for scalar mode perturbations of a scalar field coupled to gravity in Lorentzian spacetime is derived in [11]. Gauge degrees of freedom are removed, so that the action may be written in terms of one physical scalar variable q in the form:

$$iS_{2,S} = -\frac{i}{2}\int \mathrm{d}t\mathrm{d}^3x \, \sqrt{\gamma}\left((\Delta_3+3)q\left(\hat{O}' + \Delta_3 + 3\right)q\right) \tag{30}$$

where:

$$\hat{O}' = -\frac{d}{dt^2} + \frac{1}{2}\left(\phi_0'\right)^2 + \phi_0'\left(\frac{1}{\phi_0'}\right)'' \tag{31}$$

From this formula, we may obtain the Euclidean action for an axion field coupled to gravity by Wick rotating time and analytically continuing $\phi_0 \to i\phi_0$ and $\Delta_3 \to -\Delta_3$. We then find:

$$S_E = \frac{1}{2}\int d^4x \, \sqrt{\gamma}\left((-\Delta_3+3)q\left(\hat{O} - \Delta_3 + 3\right)q\right) \tag{32}$$

where

$$\hat{O} = -\frac{d^2}{d\tau^2} - \frac{1}{2}(\phi_0')^2 + \phi_0'\left(\frac{1}{\phi_0'}\right)'' \tag{33}$$

We plug in our value for the background field to find:

$$\hat{O} = -\frac{d^2}{d\tau^2} + 4 - 6\text{sech}^2(2t) \tag{34}$$

Again, $\hat{O}$ and $-\Delta_3$ commute, so we may compute their spectra separately and add them. We know that $-\Delta_3$ has no negative eigenvalues, so we will search for negative eigenvalues of $\hat{O}$. As in the case of the tensor modes, our boundary conditions are that the eigenfunction approaches zero as $\tau \to \pm\infty$.

Our differential equation is:

$$\left(-\frac{d^2}{d\tau^2}+4-6\operatorname{sech}^2(2t)\right)f=\lambda f \tag{35}$$

First, we perform a change of variables to $\tau' = 2\tau$. We then have (moving the constants into the eigenvalue):

$$\left(-\frac{d^2}{d\tau'^2}-\frac{3}{2}\operatorname{sech}^2(\tau')\right)f=\lambda' f \tag{36}$$

where $\lambda' = \lambda/4 - 1$. This differential equation has an analytical solution given by:

$$f(\tau';\lambda') = c_1 P^{i\sqrt{\lambda'}}_{\frac{1}{2}(\sqrt{7}-1)}(\tanh\tau') + c_2 Q^{i\sqrt{\lambda'}}_{\frac{1}{2}(\sqrt{7}-1)}(\tanh\tau') \tag{37}$$

where P^b_a is the associated Legendre polynomial of the first kind and Q^b_a is the associated Legendre polynomial of the second kind. We now expand f about $\tau' = \infty$ to enforce our boundary condition. To simplify notation, let $\lambda' = -\Lambda$, where $\Lambda > 0$, and let $x \equiv \tanh\tau'$. Note that $x \to 1$ as $\tau' \to \infty$. If we expand about $x = 1$, we find that:

$$f(x,\Lambda) = c_2\left[\frac{i\pi^2 2^{\frac{\sqrt{\Lambda}}{2}} e^{\frac{3}{2}i\pi\sqrt{\Lambda}}\sec\left(\frac{1}{2}\pi\left(\sqrt{7}-2\sqrt{\Lambda}\right)\right)}{\left(-1+e^{2i\pi\sqrt{\Lambda}}\right)\Gamma\left(1-\sqrt{\Lambda}\right)\Gamma\left(\sqrt{\Lambda}-\frac{\sqrt{7}}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\left(1+\sqrt{7}\right)+\sqrt{\Lambda}\right)}\right](x-1)^{-\sqrt{\Lambda}/2}+O(1) \tag{38}$$

To enforce our boundary condition, we demand that the coefficient of the leading order term approach zero as $x \to 1$. The coefficient term in the bracket, which we call $K_1(\Lambda)$, has no zeros for positive Λ (see fig. 5), and we therefore conclude that $c_2 = 0$. With this assumption, we expand f about $x = -1$ to find:

$$f(x,\Lambda) = c_1\left[\frac{\pi 2^{\frac{\sqrt{\Lambda}}{2}}\csc\left(\pi\sqrt{\Lambda}\right)}{\Gamma\left(1-\sqrt{\Lambda}\right)\Gamma\left(\sqrt{\Lambda}-\frac{\sqrt{7}}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\left(1+\sqrt{7}\right)+\sqrt{\Lambda}\right)}\right](x+1)^{-\sqrt{\Lambda}/2}+O(1) \tag{39}$$

This coefficient function, which we call $K_2(\Lambda)$, has exactly one zero (see fig. 5), given by $\Lambda = \frac{1}{2}(4-\sqrt{7}) \approx 0.68$. This then gives us that the single negative eigenvalue of $\hat{O}$ with our boundary conditions is:

$$\lambda = -4\Lambda + 4 = -\frac{1}{2}(4-\sqrt{7}) + 4 \approx -0.29 \tag{40}$$

Now we consider the other terms in our second order action. The constant $+3$ pushes up the spectrum so that all eigenvalues are positive. The scalar Laplacian on the 3-sphere has eigenvalues $l(l+2)$ beginning at $l = 0$, which are all non-negative [10]. Therefore we see that the second order action for scalar modes has a strictly positive spectrum, and we have proven that there are no negative modes about the axion wormhole instanton.

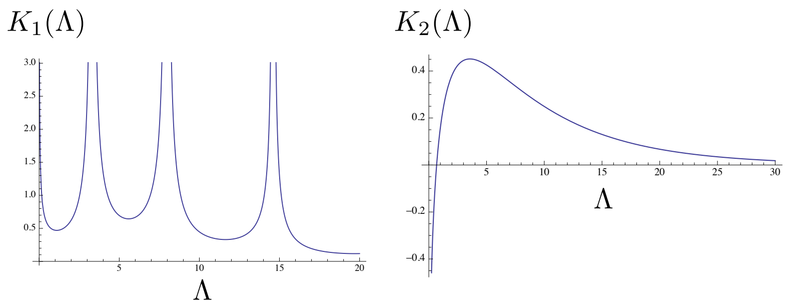

Figure 5. Plots of the coefficient functions used in the determination of boundary condition implications for the eigenfunction. The coefficient $K_1(\Lambda)$ has no zeros for positive Λ, forcing the constant $c_2 = 0$. The coefficient $K_2(\Lambda)$ has exactly one zero for positive Λ, corresponding to the lone negative eigenvalue of $\hat{O}$.

VII. DISCUSSION

When the axionic wormhole instanton was first discovered, theorists worried that the instanton represented an instability of the universe, which would cause parts of the universe to split off as disconnected baby universes, carrying their contents with them and ruining conservation laws [3]. These fears were premature, as the spectrum of the instanton had not been computed. In [13], a counterargument was presented to these views, and it was conjectured (but not proved) that the wormhole instanton has no negative modes. Our work provides the evidence to defend this claim.

A previous study of the axionic wormhole instanton claimed to have discovered a single negative mode using the method of homogeneous truncation of the perturbations [14][15]. This means that only homogeneous perturbations were considered, and used as a proxy for the full set of perturbations. Our results show that this method did not give valid information about the instanton. This is because the truncation comes before fixing the gauge, and therefore prevents one from fixing all gauge degrees of freedom. It is therefore likely that the negative mode found by this method actually comes from one of the unfixed gauge degrees of freedom.

Because of the positivity of the spectrum about the instanton, we are led to interpret it as mediating tunneling between stable, classically degenerate ground states. This means that the quantum ground state will be a superposition of these classical ground states, as in the double well. However, to understand the true ground state, we must consider variations of our wormhole instanton. First of all, the wormhole may connect any part of one of the asymptotic regions to any part of the other, meaning that we must integrate over the location of each end of the wormhole. Second, it may connect the universes at any time, meaning that we must integrate over the instant at which the universes are connected. Finally, there are more instantons that may be thought of as multiple copies of our wormhole instanton. This includes multiple wormholes between two asymptotic regions, or multiple wormholes connecting several asymptotic regions. These multiple-wormhole solutions must also be integrated over space and time. When we consider the contribution of all of these instantons, the ground state becomes a superposition of many-universe states, containing an infinite number of states with each number of universes (see fig. 6).

Although the universe is not unstable with this interpretation, there are still strange cosmological effects. Particles and information may tunnel from one classical ground state to another, apparently destroying conservation laws. This situation is remedied if we assume that the universe is in an equilibrium so that any conserved quantity tunnels away at exactly the same rate that it tunnels back, allowing conservation laws to be retained on average. However, the return tunneling does not need to be the same individual particle, for example, so in one asymptotic region, it may appear that a particle disappears from one location and reappears instantaneously at another location, possibly space-like separated [16]. This gives rise to nonlocal operators in the action, of the form:

$$S = \frac{1}{2}\sum_{i,j}\int \mathrm{d}x\mathrm{d}y\ O_i(x)C_{ij}O_j(y) \tag{41}$$

Luckily, when these operators are exponentiated in the path integral, which provides the only physical manifestation of these operators, we can write:

$$e^{-S} = \int \mathrm{d}\alpha_i\ e^{-\frac{1}{2}\alpha_i(C^{-1})_{ij}\alpha_j} e^{-\int \mathrm{d}x\ \alpha_i O_i(x)} \tag{42}$$

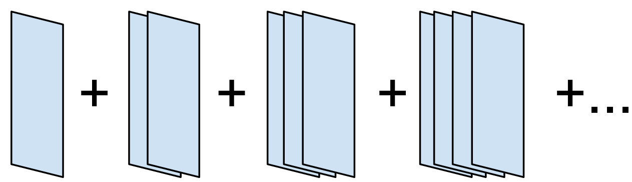

Figure 6. A schematic diagram of the ground state of the universe with an axion field coupled to gravity. The ground state is a superposition of many-universe states, which are obtained by considering many-wormhole instantons. Each term in this "sum" actually has an infinite number of terms, consisting of universes with different numbers and placements of wormholes.

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where we have introduced coupling constants α_i via a Gaussian integral. In the exponential, we see only local operators. But we have paid a price, because the α_i are integrated over, meaning that the coupling constants of nature have become random. While acceptable in principle, this result leads to paradoxical effects in the theories in which it is relevant [6].

VIII. CONCLUSIONS

We have shown that there exists a wormhole-like instanton for an axion field coupled to gravity, which mediates tunneling between classical degenerate universes. We proved that this instanton has no negative modes, so that it should be interpreted as conveying information about the superposition of the quantum ground state, as opposed to information about metastable decay. While this resolves the formerly worrisome problem about loss of quantum coherence due to axions, it raises new questions regarding randomness of the fundamental constants, which is an avenue for future research in gravitational effects of axions.

While theorists have been puzzling over these questions of axion effects, experimentalists have been at work searching for axions in our universe, notably through the ADMX project, motivated primarily by the possible role of the axion in dark matter. It is possible that we will soon discover axions in the real world, and the resulting data will give quite a different flavor to the question of the theoretical effects of axions on our universe. But now that we have shown that the axion has no negative mode, so that it cannot decay our universe, we have nothing to fear.

ACKNOWLEDGMENTS

I would like to thank Sean Hartnoll, my advisor, for guiding me in my research and for generously taking the time to walk me through advanced quantum mechanics and gravity. I would also like to thank Taylor Barrella for his advice and helpful conversations, and Julia Sommer for her support and editing.

[1] Peccei, Roberto D., and Helen R. Quinn. "CP conservation in the presence of pseudoparticles." Physical Review Letters 38.25 (1977): 1440.

[2] Kim, Jihn E. "Light pseudoscalars, particle physics and cosmology." Physics Reports 150.1 (1987): 1-177.

[3] Giddings, Steven B., and Andrew Strominger. "Axion-induced topology change in quantum gravity and string theory." Nuclear Physics B 306.4 (1988): 890-907.

[4] Coleman, Sidney. Aspects of symmetry: selected Erice lectures. Cambridge University Press, 1988.

[5] Coleman, Sidney, and Frank De Luccia. "Gravitational effects on and of vacuum decay." Physical Review D 21.12 (1980): 3305.

[6] Arkani-Hamed, Nima, Jacopo Orgera, and Joseph Polchinski. "Euclidean wormholes in string theory." Journal of High Energy Physics 2007.12 (2007): 018.

[7] Stewart, John M. "Perturbations of friedmann-robertson-walker cosmological models." Classical and Quantum Gravity 7 (1990): 1169-1180.

[8] Garriga, Jaume, et al. "Canonical quantization of cosmological perturbations in the one-bubble open universe." Nuclear Physics B 513.1 (1998): 343-374.

[9] Hertog, Thomas, and Neil Turok. "Gravity waves from instantons." Physical Review D 62.8 (2000): 083514.

[10] Higuchi, Atsushi. "Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group SO (N, 1)." Journal of mathematical physics 28.7 (1987): 1553-1566.

[11] Gratton, Steven, and Neil Turok. "Cosmological perturbations from the no boundary Euclidean path integral." Physical Review D 60.12 (1999): 123507.

[12] Gratton, Steven, and Neil Turok. "Homogeneous modes of cosmological instantons." Physical Review D 63.12 (2001): 123514.

[13] Coleman, Sidney. "Black holes as red herrings: topological fluctuations and the loss of quantum coherence." Nuclear Physics B 307.4 (1988): 867-882.

[14] Rubakov, V. A., and O. Yu Shvedov. "A negative mode about a euclidean wormhole." Physics Letters B 383.3 (1996): 258-261.

[15] Rubakov, V. A., and O. Yu Shvedov. "Instability of space-time due to Euclidean wormholes." arXiv preprint gr-qc/9608065 (1996).

[16] Giddings, Steven B., and Andrew Strominger. "Loss of incoherence and determination of coupling constants in quantum gravity." Nuclear Physics B 307.4 (1988): 854-866.