Seiberg-Witten Theory and 4-Manifolds

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1 Introduction

Today we'll talk about developments in the 80s and 90s that were revolutionary both from physical and mathematical perspectives. Timeline:

1982: Donaldson uses gauge theory to give smooth invariants for 4-manifolds

1988: Witten realizes Donaldson's invariants as the correlation functions in a TQFT (and coins that phrase)

1994: Seiberg and Witten find the exact low energy dynamics of $N = 2$ SYM and QCD

1994: Witten proposes smooth invariants equivalent to those of Donaldson

1.1 Physics Context

One of the persistent unanswered questions in physics is: what does QCD look like at low energies? For example: what causes color confinement and chiral symmetry breaking? Because this theory is asymptotically free, it becomes strongly coupled at low energy, meaning we cannot apply perturbation theory to QCD at the scales in which we are most interested.

This conundrum remains unsolved, but Seiberg and Witten were able to find a solution for a cousin theory: $N = 2$ super QCD. We actually will focus attention on the theory without matter, $N = 2$ SYM. The addition of SUSY puts constraints on a Lagrangian so that to understand the low energy dynamics, one must only find one function, the effective prepotential.

Additionally, SUSY provides a weak-strong coupling duality called S-duality that allows $N = 4$ SYM to be understood at any energy. One of the insights of Seiberg and Witten was that there is an S-duality for $N = 2$ SYM as well. The existence of this S-duality can be explained by string theory (or just by QFT classification, if you aren't comfortable with stringy arguments). There is a 6D $N = (2, 0)$ CFT that is morally a higher gauge theory of a self-dual 3-form. String theory explains the existence of this CFT as the worldvolume theory of a stack of M5-branes. The way that this relates to SYM is via compactification on a torus: the conformal symmetry in the directions of the T^2 descends as the $\mathrm{SL}(2; \mathbb{Z})$ S-duality of the effective 4D SYM.

By exploiting these benefits of $N = 2$ SUSY, Seiberg and Witten were able to understand the low energy (i.e. strong coupling) behavior of SYM and SQCD, answering the questions that we someday want to answer for non-supersymmetric QCD.

1.2 Math Context

Mathematics had a long-standing open question that looks completely unrelated: the classification of smooth structures on 4-manifolds. For dimensions less than 4, there is not enough flexibility for this question to be interesting: every topological manifold of dimension less than 4 admits exactly one smooth structure. In dimensions 4 and above, things become more interesting: some topological manifolds admit no smooth structures, and some admit many.

Note: We will assume that all of our manifolds are simply connected.

To see that something is weird with dimension 4, consider the number of inequivalent smooth structures on Euclidean space $\mathbb{R}^n$ for various n:

$$\begin{cases} 1 & n < 4 \\ \text{uncountably infinite} & n = 4 \\ 1 & n > 4 \end{cases}$$

In dimensions at least 5, there is a powerful tool for classifying smooth structures: the h-cobordism theorem (where "h" stands for homotopy equivalence). Two n-manifolds X, Y are h-cobordant if there exists an $n+1$-manifold with boundary $\bar{X} \cup Y$ such that the inclusion maps are homotopy equivalences. (Remember, all manifolds under consideration are simply connected.) For dimension at least 5, the h-cobordism theorem, which earned a Fields medal for Smale, says that h-cobordant manifolds are smoothly equivalent. This reduces the classification of smooth structures to a question of classical topology, and in particular implies that there are a finite number of distinct smooth structures on manifolds of dimension at least 5.

There is still an h-cobordism theorem in dimension 4, but it is weaker: h-cobordant manifolds are homeomorphic, but need not be diffeomorphic. The reason the stronger theorem fails in this dimension is that the proof relies on embedding 2-spheres and disentangling them, which cannot be done fully when the target is 4 dimensional.

The characterization of topological 4-manifolds was challenging, but has been understood: simply connected 4-manifolds are characterized by the intersection form on their second cohomology:

$$H^2(M) \times H^2(M) \to \mathbb{Z}$$

This form must be symmetric, and Poincare duality says that this form must be unimodular. But those are the only general requirements: Michael Freedman received a Fields medal for showing that every such form has either 1 or 2 topological 4-manifolds realizing it as their intersection form, depending on whether the form is even or odd.

2 Donaldson Invariants and Donaldson-Witten Theory

To understand smooth structures, one ideally would be able to find a “smooth invariant”, i.e. something that you can associate to a smooth 4-manifold that depends on which smooth structure is chosen. This is exactly what Donaldson was able to achieve in 1982 using gauge theory. When mathematicians talk about “gauge theory”, they are referring to the aspect that Donaldson used for his work: studying the moduli space of anti-self dual connections on bundles over the manifold. In physics language, this is the moduli space of instantons.

To remind you what an instanton means, recall that the Hodge star on a 4D vector space sends 2-forms to 2-forms:

$$\star : \Lambda^2(\mathbb{R}) \to \Lambda^2(\mathbb{R})$$

and it squares to 1, so that it has eigenvalues ± 1. Thus we may separate a 2-form into its self-dual and anti-self-dual parts:

$$\alpha = \alpha^+ + \alpha^-$$

We may extend this to all the tangent spaces of a manifold to get a map on 2-form fields:

$$\star : \Gamma\left(\Omega^2(M)\right) \to \Gamma\left(\Omega^2(M)\right)$$

The curvature of a gauge field on M is a 2-form field on M, so it may be split in this way. The instanton equation is then

$$F^+ = 0$$

This condition implies the equations of motion, because a curvature automatically satisfies the Bianchi identity $DF = 0$, and for the other:

$$D \star F = D \star F^- = -DF^- = -DF = 0$$

In fact, the instantons minimize the Yang-Mills action within their topological class.

Donaldson's idea was to study the space of solutions of this equation. One can make a topological space $\mathcal{M}_{\text{ASD}}$ out of the solutions to $F^+ = 0$, and then calculate invariants based on the topology of this space (in particular, its cohomology). These are the Donaldson invariants, and they are able to differentiate between different smooth structures on a 4-manifold. This innovative use of gauge theory was a surprise to mathematicians, and Donaldson received a Fields medal for this work.

In particular, Donaldson was able to prove a theorem excluding many 4-manifolds from admitting a smooth structure: the intersection form on a simply-connected smooth 4-manifold is diagonalizable.

In 1988, Witten described the Donaldson invariants from a different perspective: he showed that if one topologically twisted the $N = 2$ SYM, meaning that one mixes the Lorentz and R-symmetry SU(2) groups in a way that the theory becomes metric-independent, then the correlation functions of the resulting theory are the Donaldson invariants. This topological SYM is called Donaldson-Witten theory, and there has been great effort put into being able to compute these correlation functions. One successful approach was another, later insight of Witten: the Seiberg-Witten invariants.

3 Seiberg-Witten Theory

Now we return to physics for a while and sketch how Seiberg and Witten were able to explicitly calculate the low energy effective theory of $N = 2$ SYM. The classical potential of the theory is

$$V(\phi) = \frac{1}{g^2} \mathrm{Tr} \left[\phi, \phi^\dagger\right]^2$$

where ϕ is the scalar field in the vector multiplet, often called the Higgs field. This is minimized by, for SU(2) gauge group,

$$\phi = \begin{pmatrix} a & \\ & -a \end{pmatrix}$$

But we also need to consider the action of the gauge group, which takes $a \leftrightarrow -a$, so that the gauge-invariant quantity that parameterizes the space of vacua is $u = \frac{1}{2}a^2 = \mathrm{Tr}\phi^2$. Classically, for non-zero u, the gauge symmetry is spontaneously broken to $U(1)$. For $u \to \infty$, the effective coupling is small, and one may compute with semiclassics. Classically, the gauge symmetry would be unbroken at $u = 0$, but this is not the case quantum mechanically, where the gauge symmetry is broken everywhere on the u-plane.

Using S-duality, Seiberg and Witten associated to each point on the u-plane an elliptic curve, and were able to write the low-energy dynamics of the theory in terms of this curve. There are three points where this curve degenerates, and these are the three points at which the theory is weakly-coupled, if one uses the right variables. For $u = \infty$, this is the original variables, while at the other points, $u = \pm\Lambda$, there are monopole and dyon solitonic states becoming light.

4 Seiberg-Witten Invariants

A couple of months after the Seiberg-Witten solution of $N = 2$ SYM and SQCD, Witten published another paper showing how this solution could be applied to his earlier idea of Donaldson-Witten theory. Remember that Donaldson-Witten theory was the topological twist of $N = 2$ SYM, so Witten considered the topological twist applied to the low-energy effective theory as well. Because the Seiberg-Witten solution only goes up to two derivatives, it would provide only an approximate solution to a physical theory, with accuracy depending on the physical scale Λ. But the topologically twisted theory is scale invariant, so that the twisted Seiberg-Witten solution should be exact, and allow us to calculate the Donaldson invariants in an easier way. In other words, the topological nature of Donaldson-Witten theory allows us to calculate the correlation functions at short-distance or long-distance and find the same answer.

If one topologically twists the low energy effective theory near $u = \pm\Lambda$, then the partition function of the theory will count solutions to the monopole equations known as the Seiberg-Witten equations:

$$\begin{cases} D^A\psi = 0 \\ F_A^+ = \sigma(\psi) \end{cases}$$

The map $\sigma : W^+ \to i\Lambda^2_+$ is the "squaring map" that can be defined by

$$\sigma(\psi) = \psi \otimes \psi^* - \frac{1}{2} |\phi|^2 \,\mathrm{id}$$

Explicitly, the equation looks like:

$$\begin{cases} \frac{1}{2}(F_{12} + F_{34}) = |M_1|^2 - |M_2|^2 \\ \frac{1}{2}(F_{13} + F_{42}) = i\left(M_1 M_2^* - M_1^* M_2\right) \\ \frac{1}{2}(F_{14} + F_{23}) = M_1 M_2^* + M_1^* M_2 \end{cases}$$

To show that these invariants are related, we choose a metric g_0 on our 4-manifold, and look at the family tg_0 for $t \to \infty$ and $t \to 0$. The correlation functions of the topological theory are metric-independent, so won't depend on t. For $t \to 0$, the theory becomes weakly coupled, and the classical description of the theory is valid, so that we recover the Donaldson invariants as the correlation functions. In particular, this means that for a theory without abelian instantons, the u-plane integral only gets a contribution from $u = 0$, which is the SU(2) theory.

For $t \to \infty$, our theory is described well by the quantum vacuum states on $\mathbb{R}^4$. This means that the Donaldson-Witten partition function should look like:

$$Z_{DW} = Z_u + Z_{u=1} + Z_{u=-1}$$

where the first term is the contribution from the non-singular points, and the latter terms are extra contributions from the points with extra massless particles. But, Moore and Witten showed that the non-singular part vanishes for manifolds with $b_2^+ > 1$. This then gives rise to Witten's "magic formula" relating the Donaldson and Seiberg-Witten invariants in simple cases.

Now let's talk about the invariants themselves. The Seiberg-Witten invariants then come from integrating a cohomology class over the moduli space of solutions. We call a 4-manifold of "simple type" if all of its moduli spaces are 0-dimensional, in which case this integral reduces to a signed-count of solutions up to gauge equivalence. It is conjectured that all simply connected 4-manifolds are of simple type, so that we can define the Seiberg-Witten invariants as:

$$\mathrm{SW}_M : \left\{\mathrm{spin}^{\mathbb{C}} \text{ structures on } M\right\} \to \mathbb{Z}$$

defined by

$$\mathrm{SW}_M(s) = \#\left(\{\text{solutions of Seiberg-Witten equations}\}/\mathcal{G}\right)$$

Perhaps further study of these invariants and their generalizations will lead to even more Fields medal-winning work.

References

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- Labastida and Mariño, *Topological Quantum Field Theory and Four Manifolds*