The Supersymmetric Proof of the Index Theorem

Alex Kinsella

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1 Introduction: The Index Theorem for Dirac Operators

Today I'll talk about a proof of one version of the Atiyah-Singer index theorem using supersymmetric quantum mechanics.

The Atiyah-Singer index theorem beautifully straddles geometry, topology, and analysis, and unifies many classical theorems, such as the Gauss-Bonnet theorem, the Hirzebruch signature theorem, and the Riemann-Roch theorem. To begin, we will review the setting of the theorem and discuss a few examples.

The analytical quantity is the *index of an operator*, which we will now introduce. Suppose we have a linear operator $A : V \to W$, where V and W are vector spaces that may be infinite dimensional. We often want to know about existence and uniqueness of solutions to equations of the form $Av = w$ for a given $w \in W$. The relevant information is contained in $\ker A$ and $\text{coker}A = W/\text{im}A$, which tell how much our operator fails to be injective and surjective. We will restrict to a particularly nice class of operators: elliptic operators on compact manifolds. I do not want to give the definition of ellipticity here, but some familiar examples include the Laplacian, Dirac operator, and exterior derivative. The property of elliptic operators we will be interested is that they are *Fredholm* (on compact manifolds), which means that they have finite dimensional kernels and cokernels. Thus, in theory, we can get a handle on $\ker A$ and $\text{coker}A$. One would hope that at least we could determine the dimensions of these spaces. But, unfortunately, the dimensions of the kernel and the cokernel of an elliptic operator are generally hard to find, in part because they are sensitive to perturbations: small changes in parameters defining the operator may cause these dimensions to jump up and down. However, there is a quantity that is not sensitive to perturbations, and that is the *index:*

$$\text{Ind}A = \dim\ker A - \dim\text{coker}A$$

The index doesn't tell us all that we'd want to know about solving $Av = w$, but it gives us some information that is insensitive to perturbations. For example, when the index is nonzero, we know that either the kernel or the cokernel is nonempty. The Atiyah-Singer index theorem says that this analytical index, which contains information about solving operator equations, is equal to a topological index which depends only on the topology of the space on which the operator is defined. The theorem holds quite generally, but we will only look at special cases today.

In the finite dimensional case, the index is determined by the dimensions of the vector spaces involved. This is the content of the Rank-Nullity theorem, which is familiarly stated as

$$\dim\ker A + \dim\text{im}A = \dim V$$

But we can write $\dim \mathrm{im} A = \dim W - \dim \mathrm{coker} A$ and find

$$\mathrm{Ind} A = \dim V - \dim W$$

In particular, when V and W are the same dimension (i.e. when A is representable by a square matrix), A must have index zero. The Rank-Nullity theorem can be considered the simplest instance of the Atiyah-Singer index theorem, as it equates the analytical index to a topological quantity, the dimensionality of the vector spaces. The other familiar instance of the index theorem is the Gauss-Bonnet formula, which says that for a compact 2-dimensional manifold M, the Euler characteristic is the integrated Gauss curvature:

$$\chi(M) = \frac{1}{2\pi} \int_M K \, dA$$

Interpreting this case is somewhat backwards from what one might expect: the Euler characteristic is the analytic side of this equation, as it is the index of the exterior derivative: $\mathrm{Ind} d = \chi(M)$, while the right hand side is the topological side, as it is the Euler class of TM written via the Chern-Weil homomorphism.

In physics, we are interested in operators that act on infinite dimensional vector spaces. In particular, we are interested in differential operators such as the Laplacian or Dirac operator that act on field configurations. Such a differential operator D is a linear map on smooth sections of vector bundles E, F over a spacetime M:

$$D : \Gamma(M, E) \to \Gamma(M, F)$$

For example, the Laplacian acts on smooth functions (scalar field configurations) on M:

$$\Delta : C^\infty(M) \to C^\infty(M)$$

When the bundles E and F have inner products, as they always do in physics, we may define an adjoint operator

$$D^\dagger : \Gamma(M, F) \to \Gamma(M, E)$$

that acts in the opposite direction and satisfies

$$\langle De, f \rangle_F = \left\langle e, D^\dagger f \right\rangle_E$$

This gives us a different expression for the index in terms of the adjoint operator. First we notice that $\ker D^\dagger \cong (\mathrm{im} D)^\perp$. To see this, suppose $f \in \ker D^\dagger$ and $f' \in \mathrm{im} D$, so $f' = De'$ for some $e' \in \Gamma(M, E)$. Then we have

$$\langle f', f \rangle_F = \langle De', f \rangle_F = \left\langle e', D^\dagger f \right\rangle_E = 0$$

Conversely, suppose $g \in (\mathrm{im} D)^\perp$, and consider $D^\dagger g$. We take an inner product with arbitrary $e \in \Gamma(M, E)$ and find

$$\left\langle e, D^\dagger g \right\rangle_E = \langle De, g \rangle_F = 0$$

so we are done. In the case of finite dimensions, this isomorphism tells us that

$$\dim \ker D^\dagger = \dim (\mathrm{im} D)^\perp = \dim \Gamma(M, F) - \dim \mathrm{im} D = \dim \mathrm{coker} D$$

In infinite dimensions, one needs a different proof, but the same result holds. Thus we may write the index as

$$\text{Ind}D = \dim\ker D - \dim\ker D^\dagger$$

We should show that this quantity is stable against perturbations, but we will instead give a similar proof of a physical index later, because the ideas are the same but the language is more familiar.

Now we will derive an alternate formula for the index that will provide our bridge to physics. Define a bundle $G = E \oplus F$ and operators on G by

$$\begin{aligned}\tilde{Q} &= \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \\ \tilde{H} &= \tilde{Q}^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger \end{pmatrix} \\ \Gamma &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Then another formula for the index is

$$\begin{aligned}\text{Ind}D &= \text{Tr}e^{-\beta D^\dagger D} - \text{Tr}e^{-\beta DD^\dagger} \\ &= \text{Tr}\Gamma e^{-\beta\tilde{H}}\end{aligned}$$

for any $\beta > 0$. To prove this, we first want to show that $\ker D = \ker D^\dagger D$. To show the nontrivial direction, suppose $e \in \ker D^\dagger D$. Then we have

$$\left\langle D^\dagger De, e'\right\rangle_E = 0$$

for all $e' \in \Gamma(M, E)$. Then

$$\left\langle De, De'\right\rangle_F = 0$$

for all $e' \in E$. In particular, $\langle De, De\rangle_F = 0$, which implies that $De = 0$, so that $e \in \ker D$. So we are done. Similarly, $\ker D^\dagger = \ker DD^\dagger$. Next, let $\{\phi_n\}$ be an orthonormal set of eigensections of $D^\dagger D$ with eigenvalues λ_n. Then for each ψ_n with positive eigenvalue, we can define $\psi_n = D\phi_n/\sqrt{\lambda_n}$, which is an eigensection of $DD^\dagger$ with the same eigenvalue. These are orthonormal eigensections as well. Additionally, let $\left\{\phi_i^0\right\}$ and $\left\{\psi_i^0\right\}$ be orthonormal eigensections spanning $\ker D$ and $\ker D^\dagger$. Then we may compute

$$\begin{aligned}\text{Tr}e^{-\beta D^\dagger D} - \text{Tr}e^{-\beta DD^\dagger} &= \sum_{\lambda_n\neq 0}\langle\phi_n|e^{-\beta D^\dagger D}|\phi\rangle - \sum_{\lambda_n\neq 0}\langle\psi_n|e^{-\beta DD^\dagger}|\psi_n\rangle + \\ &\quad + \sum_i\langle\phi_i^0|\phi_i^0\rangle - \sum_j\langle\psi_j^0|\psi_j^0\rangle \\ &= \sum_{\lambda_n\neq 0} e^{-\beta\lambda_n}\left(\langle\phi_n|\phi_n\rangle - \langle\psi_n|\psi_n\rangle\right) + \sum_i 1 - \sum_j 1 \\ &= \dim\ker D - \dim\ker D^\dagger\end{aligned}$$

so we are done. We will see later a reprise of this argument in the physical context.

Now let's move on to the case of interest. Let M be an even dimensional spin manifold. Because we are in even dimensions, the spinor bundle splits into even and odd chirality parts:

$$\not{S} = \not{S}_+ \oplus \not{S}_-$$

with the splitting accomplished by projecting to the ± 1 eigenspaces of the chirality operator γ_5. Then the Dirac operator $i\not\nabla : \Gamma(M, \not{S}) \to \Gamma(M, \not{S})$ may be written in block diagonal form as

$$i\not\nabla = \begin{pmatrix} 0 & \not{D}^\dagger \\ \not{D} & 0 \end{pmatrix}$$

where $\not{D} : \Gamma(M, \not{S}_+) \to \Gamma(M, \not{S}_-)$ is the *chiral* Dirac operator. Then we may consider the index of the chiral Dirac operator

$$\mathrm{Ind}\not{D} = \dim\ker \not{D} - \dim\ker \not{D}^\dagger$$

Physically, this is measuring the difference in the number of left handed and right handed spinors solving the massless Dirac equation.

The Atiyah-Singer index theorem tells us that this index is equal to a topological index, given by the integral of the $\hat{A}$ genus, an element of cohomology of the underlying manifold:

$$\mathrm{Ind}\not{D} = \int_M \hat{A}(TM)$$

The $\hat{A}$ genus is a characteristic class of the tangent bundle, meaning that it is a topological invariant of the underlying manifold. It can be defined via the Chern-Weil homomorphism as

$$\hat{A}(TM) = \det \sqrt{\frac{\mathcal{R}/4\pi}{\sinh \mathcal{R}/4\pi}}$$

where $\mathcal{R}$ is the curvature two-form, i.e. the $\mathfrak{so}(n)$-valued 2-form on M that we typically represent with the Riemann curvature tensor. (Remember that, although this looks like it depends on the connection through the curvature, it is actually a topological invariant that gives the same cohomology class for any connection put on the bundle.) Our job will be to reproduce this formula via supersymmetric quantum mechanics.

2 Supersymmetric Quantum Mechanics

Our strategy for proving the index theorem for the spin complex on M is to to probe the manifold with a spinning particle i.e. to study a 0+1 sigma model with target M. It turns out that such a theory will always have worldline supersymmetry, whether or not the effective target space theory is supersymmetric. The worldline supercharge of the theory may be identified with an operator on M, and the index may then be computed via a path integral. Which version of the index theorem is relevant depends on the details of the sigma model under consideration. The spinning particle, which gives the result for the spin complex on M, is the $N = 1/2$ sigma model. There are three steps to the proof of the index theorem:

1. Identify the index with a physical quantity that may be computed by a path integral
2. Reduce the path integral to a functional determinant
3. Compute the determinant

First we will set up the physical theory. Let $\mathcal{W}$ be a 1-manifold representing the worldline of our spinning particle. We consider embeddings into Euclidean spacetime M, given by a bosonic embedding field X and a

Grassman fermionic "embedding field" ψ, which assigns spins along the worldline. Having $N = 1/2$ instead of $N = 1$ SUSY means that we have just one fermion instead of two, as the superparticle would have.

The Lagrangian for our particle moving and spinning in M is then

$$L = \frac{1}{2}g_{ij}(X)\frac{dX^i}{dt}\frac{dX^j}{dt} + \frac{i}{2}g_{ij}(X)\psi^i\frac{D\psi^j}{dt}$$

where t is the coordinate on $\mathcal{W}$ and D is the covariant derivative on M. The first term is familiar from bosonic sigma models, while the second term gives the spin dynamics. Despite not seeking out SUSY, this action has a worldline supersymmetry generated by

$$\begin{aligned} \delta_\epsilon X^j &= i\epsilon\psi^j \\ \delta_\epsilon \psi^j &= -\epsilon\frac{dX^i}{dt} \end{aligned}$$

where ϵ is an infinitesimal real Grassman parameter. We may use Noether's theorem to compute the supercharge for this symmetry:

$$Q = ig_{ij}(X)\frac{dX^i}{dt}\psi^j$$

The relevance to our index problem can be seen from canonically quantizing this system. The bosonic embedding field is quantized in the standard way:

$$\begin{aligned} X &\mapsto \hat{X} \\ P &\mapsto i\nabla \end{aligned}$$

The Poisson bracket for the fermionic field is promoted to the anticommutator

$$\left\{\psi^i, \psi^j\right\} = g^{ij}$$

This is the Clifford algebra for M, so that quantization of these fields consists of representing them by the gamma matrices on M:

$$\psi^i \mapsto \gamma^i/\sqrt{2}$$

The Hamiltonian is

$$H = g_{ij}(X)\dot{X}^i P^j - g_{ij}(X)\dot{\psi}^i\frac{i}{2}\psi^j - L = -\frac{1}{2}\nabla\!\!\!/^2$$

where Δ is the Laplacian on M. This Hamiltonian acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2^n}$. The quantization of the supercharge is

$$Q \mapsto ig_{ij}(X)\frac{\gamma^i}{\sqrt{2}}\nabla^j = \frac{1}{\sqrt{2}}\nabla\!\!\!/$$

where $\nabla\!\!\!/$ is the Dirac operator on M. So we have succeeded in constructing a theory where the supercharge is the operator of interest on M. From here, it is easy to verify that the SUSY algebra is satisfied:

$$\{Q, Q\} = 2QQ = \nabla\!\!\!/^2 = -2H$$

as desired.

3 Path Integral Form for the Index

Now we will investigate the manifestation of the index of the Dirac operator in SQM and find a path integral formula that computes it. In Witten's paper "Constraints on Supersymmetry Breaking", he coined an operator that diagnoses whether or not a supersymmetric system can undergo spontaneous breaking of SUSY. The operator is $(-1)^F$, which acts as 1 on bosonic states and -1 on fermionic states. (Note that a concrete realization of this operator is $\exp(2\pi i J_z)$, but we won't make use of this form.) We will be interested in the trace of this operator. The important fact is that only zero energy states contribute to the trace, because states with nonzero energy are paired by the action of Q and therefore cancel in the trace. To see this, let b be a bosonic state of non-zero energy E. Then define the fermionic state $f = \frac{1}{\sqrt{E}} Qb$. These are then paired as

$$Qb = \sqrt{E} f; \quad Qf = \sqrt{E} b$$

So all states with nonzero energy are paired in two-dimensional multiplets with the same energy. However, this is not the case for zero-energy states, because they are annihilated by the supercharge. To see this, suppose $Hv = 0$. Then $\langle v, Hv \rangle = 0$ implies $\langle v, Q^2 v \rangle = 0$, and skew-self-adjointness of Q gives $\langle Qv, Qv \rangle = 0$, so that $Qv = 0$. So states of zero energy are in one-dimensional supersymmetry multiplets, and are thus unconstrained. When we calculate the trace of $(-1)^F$, we find

$$\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}$$

which is the difference in the number of boson and fermionic zero-energy states. This quantity is stable to perturbations, by an argument that, with minor adjustments in formulation, applies as well to the index of elliptic operators. The idea is that suppose a perturbation were to change the value of this trace by changing energies from zero to nonzero and vice-versa. As soon as a state leaves zero energy, it must have a supersymmetric partner - thus, a perturbation must lift an equal number of bosonic and fermionic zero modes. Similarly, nonzero-energy states must reach zero energy in boson-fermion pairs. Either way, the trace remains constant.

As an aside, Witten's motivation for introducing this quantity is that if $\mathrm{Tr}(-1)^F$ is nonzero, there exists at least one zero-energy state, which must be supersymmetric, as seen above, so that spontaneous supersymmetry breaking is not possible.

The trace above is not absolutely convergent as written, so the cancellation we just sketched does not make sense. The solution is to regulate the high energy states, which Witten suggests doing via

$$\mathrm{Tr}(-1)^F \equiv \mathrm{Tr}(-1)^F e^{-\beta H}$$

for some positive β. The trace is independent of the chosen value of β, which is what will give us so much power in our path integral. Now we may reinterpret this trace: we may think of it as a partition function with boundary conditions twisted by $(-1)^F$.

Now we are in the position to write the index of the Dirac operator in terms of physical quantities. We have our earlier formula that

$$\begin{aligned}\text{Ind}\not{D} &= \text{Tr}\Gamma e^{-\beta\tilde{H}} \\ &= \text{Tr}\Gamma e^{-\beta\tilde{Q}^2} \\ &= \text{Tr}\Gamma e^{-\beta\not{\nabla}^2} \\ &= \text{Tr}(-1)^F e^{-2\beta Q^2} \\ &= \text{Tr}(-1)^F e^{-\beta H}\end{aligned}$$

In the first two lines, we have the tilde operators that were defined for any Fredholm operator, and in the third line we specialize to the Dirac operator. Then we invoke our quantization $Q \mapsto \frac{1}{\sqrt{2}}\not{\nabla}$ and recognize that $(-1)^F$ has the same action on the Hilbert space as Γ does on the space of sections.

Now we have the path integral formula for the index

$$\text{Ind}\not{D} = \text{Tr}(-1)^F e^{-\beta H} = \int_{\text{PBC}} \mathcal{D}x\mathcal{D}\psi\ e^{-\int_0^\beta dtL}$$

The periodic boundary conditions on the fermions is the effect of including $(-1)^F$ in the integral. Normal boundary conditions on the fermions are antiperiodic due to their anticommuting nature. The key to evaluating this integral is to note that the index is independent of β, so the semiclassical $\beta \to 0$ limit of the path integral must be exact. In other words, only the contributions to the path integral coming from classical solutions and the second order fluctuations about them are independent of β, and thus these terms must exactly give the index.

First, let's find the relevant classical solutions. We rescale our parameterization of the worldline via $t = \beta s$ to get the action

$$S = \int_0^1 ds\ \left[\frac{1}{\beta}\frac{1}{2}g_{ij}(X)\frac{dX^i}{ds}\frac{dX^j}{ds} + \frac{1}{2}g_{ij}(X)\psi^i\frac{D\psi^j}{Ds}\right]$$

As we take $\beta \to 0$, we see that any path with $\frac{dX}{ds} \neq 0$ will contribute an exponentially small amount to the path integral. So we only need to consider paths where X is constant. The Euler-Langrange equation for the fermion is

$$\frac{D\psi^j}{Ds} = 0$$

which on our constant path reduces to ψ being constant as well. So the relevant classical solutions are those with $X = X_0$ and $\psi = \psi_0$. These solutions have zero action, so they do not contribute to the path integral.

So the index reduces to the quadratic fluctuations. To compute these, we expand our fields as

$$\begin{aligned}X^j(s) &= X_0^j + \xi^j(s) \\ \psi^j(s) &= \psi_0^j + \eta^j(s)\end{aligned}$$

The quadratic action is

$$S_2 = -\frac{1}{2\beta}\int_0^1 ds\ \left[\left|\dot{\xi}\right|^2 + \langle\eta,\dot{\eta}\rangle - \frac{1}{2}\psi_0^i\psi_0^j R_{ijkl}\xi^k\dot{\xi}^l\right]$$

where R_{ijkl} is the Riemann curvature. The bosonic and fermionic perturbations are not mixed in this action, so that the action separates as

$$S_2 = I_1(X_0, \psi_0) + I_2$$

with

$$I_1(X_0, \psi_0) = -\frac{1}{2\beta} \int_0^1 ds \ \left(\left| \dot{\xi} \right|^2 - \frac{1}{2} \left\langle \xi, \psi_0^i \psi_0^j R_{ij} \dot{\xi} \right\rangle \right)$$

and

$$I_2 = -\frac{1}{2\beta} \int_0^1 dt \ \langle \eta, \dot{\eta} \rangle$$

Notice that the fermionic integral is independent of the chosen constant solution.

The index separates as

$$\text{Ind} \not{D} = \int \mathcal{D}\xi \mathcal{D}\eta \ e^{-S_2} = \int \mathcal{D}\xi \ e^{-I_1} \int \mathcal{D}\eta \ e^{-I_2}$$

The fermionic integral is independent of the chosen constant solution, so it contributes a multiplicative constant to the index (given by the Pfaffian of $\frac{d}{ds}$), which we will denote C^{2n}. So the index reduces to computation of the bosonic part:

$$\text{Ind} \not{D} = C^{2n} \int \mathcal{D}\xi \ e^{-I_1}$$

We can rewrite the quadratic action as

$$I_1 = -\frac{1}{2} \int_0^1 \langle D_B(\xi), \xi \rangle$$

where

$$D_B(X_0, \psi_0) = -\frac{d^2}{ds^2} - \frac{1}{2} \psi_0^i \psi_0^j R_{ij} \frac{d}{ds}$$

So for any chosen constant solution, we end up with a determinant of D_B. We must integrate over all constant solutions, so that

$$\text{Ind} \not{D} = C^{2n} \int_{\Pi TM} \frac{1}{\sqrt{\det D_B(x, \psi)}}$$

where the integral over ΠTM means that we integrate over the supermanifold of constant solutions. Integration over the supermanifold sounds hard, but luckily we can reduce this to integration over M instead: the idea is that functions on ΠTM may be reinterpreted as differential forms on M. We won't go through the details of this, but the result is that our integral becomes

$$\text{Ind} \not{D} = (C')^n \int_M \frac{1}{\sqrt{\det D_B(x, \psi)}}$$

where the ψ dependence in the integral turns the integrand into a differential form of mixed degree. Say we replace $\psi_0^i \psi_0^j R_{ij}$ by an element $g \in \text{so}(TY_{X_0})$. Then one may evaluate the functional determinant and find

$$\sqrt{\det D_g} = C^{2n} \left(\frac{g/2}{\sinh g/2} \right)^{-1}$$

And therefore

$$(\det D_g)^{-1/2} = C^{2n} \hat{A}(g)$$

(where we are being sloppy with the constant factor). Thus we have

$$(\det D_B)^{-1/2} = C^{2n}\hat{A}(R)$$

where R is the curvature of M. Which gives the desired result

$$\mathrm{Ind} \not{D} = \int_M \hat{A}(R)$$

where we have fixed the constant to the known value (perhaps by evaluating on a simple example).

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