BRST, Gauge Theory, and Cohomological Field Theory

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January 31, 2018

1 Introduction

Today I'm going to talk about BRST cohomology and how it relates to gauge theory and cohomological field theory. The original inspiration for this talk was trying to understand the relation between BRST cohomology in gauge theories like the Standard Model and the "BRST-like" operators in topological sigma models, or more generally, in cohomological field theories. We will begin with a brief description of equivariant cohomology, the mathematical idea that unifies the two apparently different roles of BRST operators in physics. Then we'll talk about BRST cohomology in classical gauge theory, followed by the role of BRST in cohomological field theories.

I'll quickly review the idea of (co)homology because it will be central to the talk. Say we have a sequence A^i of abelian groups along with group homomorphisms $d_i : A^i \to A^{i+1}$ that satisfy $d_i \circ d_{i-1} = 0$. (Often we suppress the index on d_i.) Then We call (A^i, d_i) a cochain complex. The condition on d_i tells us that $\mathrm{im} d_{i-1} \subset \ker d_i$. The failure of inclusion in the other direction is called the cohomology of the complex:

$$H^i(d) = \ker d_i / \mathrm{im} d_{i-1}$$

We call it "homology" instead of cohomology when the differential decreases the degree by one instead of increasing it by one. In this case we denote it by ∂_i and call it a boundary. We also lower the index on the complex and homology to A_i and H_i.

The most familiar examples of this algebraic constructions are the homology and cohomology of smooth manifolds. The deRham cohomology comes from the the complex of differential forms $(\Omega^*(M), d)$ with the exterior derivative, while the singular homology comes from the complex of chains $(C^*(M), \partial)$ with the boundary map.

It is common that a cohomology theory has many "models", i.e. many chain complexes that share the same homology. For example, the cohomology of a smooth manifold may be calculated either by the deRham complex we just discussed, or by the complex of singular cochains, which looks superficially quite different. The equivariant cohomology we discuss below has many model cochain complexes, several of which are popular in the literature, which means translation is sometimes required.

2 Unifying Principle: Equivariant Cohomology

I wanted to give this talk because I wanted to understand the link between BRST methods in gauge theories and in cohomological field theory. It turns out that the unifying idea is "equivariant cohomology". The idea is to compute the cohomology of not just a manifold, but a manifold with an action by a Lie group G.

The simplest definition of equivariant cohomology would be as the cohomology of the quotient space:

$$H_G^*(M) \stackrel{?}{=} H^*(M/G)$$

The problem is that when the action of G on M has fixed points, i.e. when the action is not free, M/G will have singularities, making it difficult to define cohomology of the quotient space. The solution to this problem is to modify the space so that the action becomes free.

To do so, we will use the universal bundle EG of G. This is a special topological space that is contractible and acted upon freely by G. This gives EG the structure of a G-principal bundle $\pi : EG \to BG$, and the base BG of this bundle is called the classifying space for G. The space BG is unique up to homotopy type, and the space EG is unique up to equivariant homotopy type.

Because G acts freely on EG, this lets us modify our G-action on M to remove the fixed points: we'll consider instead the diagonal G-action on $EG\times M$, which is guaranteed to be free. Thus the quotient of $EG\times M$ by G is a manifold. We'll denote this space as $EG \times_G M$. Because EG is contractible, $EG \times M$ is homotopy equivalent to M, so the modification we have made will not affect cohomology. We define the topological equivariant cohomology of $G \circlearrowright M$ to be

$$H^*_{G,\text{top}}(M) = H^*\left(EG \times_G M\right)$$

There are a few algebraic models of equivariant cohomology that give equivalent results to the topological formula just given. Three of these models are popular in the literature:

1. The Weil model, which is a cochain complex meant to algebraically represent connections and curvature on EG
2. The Cartan model, which is a cochain complex of equivariant differential forms on M with an equivariant exterior derivative
3. The BRST model, which is a sort of hybrid of the last two: it takes the cochains of the Weil model but modifies the differential so that it coincides with that of the Cartan model when restricted to the correct subspace

Of course, the BRST model is the one that arises naturally in physics applications. Now we'll move on to discuss the appearances of BRST cohomology in physics.

3 BRST in Classical Gauge Theories

First, we'll talk about the BRST method in classical gauge theories. We'll start by showing the problem we want to solve and the way that a BRST operator solves it, and then explain the relation to equivariant cohomology. The classical BRST procedure is similar to quantum gauge theories, where we are familiar with BRST quantization. Although we aren't used to seeing BRST in classical theories, it is an elegant way to isolate the gauge invariant part of the theory.

Say we have a gauge theory with gauge group G. Then the phase space P is redundant, so that the physical phase space is given by a constraint surface $\Sigma = \{g_i = 0\} \subset P$. Examples of such constraints include Gauss's law in E&M and the Hamiltonian constraint in general relativity. (We will assume throughout that our constraints are "irreducible" for simplicity.) The intuitive solution to the redundancy is to work directly with the quotient P/G. This goes by the name of "symplectic reduction", or in more general cases "coisotropic reduction". There are a few problems with the straightforward implementation of symplectic reduction in gauge theories:

1. It usually sacrifices manifest locality and Lorentz invariance
2. The quotient space may have singularities
3. It may be impossible to actually compute the quotient, as this requires explicit solution of the PDEs coming from the constraints

To sidestep these problems, we will implement an algebraic approach to symplectic reduction that goes by the name of BRST cohomology. Instead of shrinking the phase space, our solution will be to *extend* the phase space and then perform a homological operation by which the introduced degrees of freedom "cancel" the unphysical gauge degrees of freedom. Because of this, we may work with functions defined on the original phase space, so that we retain all the advantages of our original formulation. This is the BRST procedure, and the extra degrees of freedom are called ghosts and ghost momenta.

The reduction to the physical part of phase space consists of two steps:

1. Restrict from functions $C^\infty(P)$ on phase space to functions $C^\infty(\Sigma)$ on the constraint surface
2. Restrict to functions that are gauge invariant, i.e. constant along gauge directions away from the constraint surface

To accomplish the first step, we create a homological resolution of $C^\infty(\Sigma)$ by $C^\infty(P)$-modules. This means a chain complex of $C^\infty(P)$ modules such that the homology vanishes except in degree zero, where it is $C^\infty(\Sigma)$. The point of this is so that we can continue to work with functions on the entire phase space. (This is analogous to working with a presentation of a group.) So we want to create a chain complex with boundary map δ such that $H_0(\delta) = C^\infty(P)$ and $H_k(\delta) = 0$ for $k \neq 0$. We will do this not by finding a sequence of different algebras, but instead by creating a superalgebra containing $C^\infty(P)$ and giving it a grading. The relation between the two algebras of functions is fairly simple: $C^\infty(\Sigma) \cong C^\infty(P)/\mathcal{N}$, where $\mathcal{N}$ is the ideal of phase space functions that vanish on Σ. So we want to satisfy

$$H_0(\delta) = \frac{\ker \delta_0}{\mathrm{im}\delta_1} = \frac{C^\infty(P)}{\mathcal{N}}$$

thus we choose δ such that $\ker \delta_0 = C^\infty(P)$ and $\mathrm{im}\delta_1 = \mathcal{N}$. To satisfy the first condition, we choose $\delta z^A = 0$ for all phase space coordinates z^A. Then $\delta F = 0$ for any phase space function $F(z^A)$ because δ is a derivation. This means that we assign all phase space variables z^A degree zero. We will call the degree in this complex the "antighost number".

Now we want to achieve $\mathrm{im}\delta_1 = \mathcal{N}$. It is a fact that any element of $\mathcal{N}$ can be written as a linear combination of the constraint functions with coefficients that are phase space functions. Thus we will introduce generators $\mathcal{P}_i$ into our algebra, one for each constraint g_i, and declare $\delta\mathcal{P}_i = g_i$. To get the correct grading, we choose $\mathrm{antigh}\mathcal{P}_i = 1$ and we take $\mathcal{P}_i$ to be an odd element of the algebra so that δ is an odd derivation. So our algebra is $\mathbb{C}[\mathcal{P}_i] \otimes \mathbb{C}^\infty(P)$. WHY?

To enforce gauge invariance, we introduce the "longitudinal derivative" d. A vector field on the constraint surface is said to be "longitudinal" if it is everywhere tangent to the gauge orbits. These are the vector fields associated to infinitesimal gauge transformations:

$$X_i F = \{F, g_i\}$$

We define longitudinal p-forms as forms that act on longitudinal vectors. These may be represented as polynomials in the 1-forms η^i dual to the vector fields X_i, with coefficients that are smooth functions on Σ, so that the longitudinal algebra is $C^\infty(\Sigma) \otimes \mathbb{C}[\eta^i]$. We will call the η^i "ghosts". The form degree is the pure ghost number, so we have

$$\text{pure gh}(\eta^i) = 1, \quad \text{pure gh}(z^A) = 0$$

We may define a longitudinal derivative d that acts on longitudinal p-forms. On our basis of 1-forms, this looks like

$$\begin{aligned} dF &= (\partial_a F)\eta^a \\ d\eta^a &= \frac{1}{2}\omega^b\omega^c C_{cb}{}^a \end{aligned}$$

where $C_{cb}{}^a$ are the structure constants of the Lie algebra. We take cohomology of this derivative, and we have in particular that $H^0(d)$ is the space of gauge invariant functions on Σ. In fact, we may extend the longitudinal forms, as well as the derivative, to all of phase space by taking the coefficients to lie in $C^\infty(P)$. However, now we only have $d^2 \approx 0$, meaning that it is zero only modulo the constraint functions. This will be remedied by the BRST operator.

Now it's time to combine these algebras into one. We take the ghosts and ghost momenta to be canonically conjugate variables, and to have trivial bracket with the phase space variables. Then we have an extended phase space with the ghosts and momenta added, and we take the algebra of superfunctions on extended phase space as

$$\mathbb{C}[\mathcal{P}_i] \otimes C^\infty(P) \otimes \mathbb{C}[\eta^i]$$

We extend the degrees as $\text{antigh}(\eta^i) = 0$ and $\text{puregh}(\mathcal{P}_i) = 0$. Then the "ghost number" is defined for any variable on phase space as

$$\text{gh}A = \text{puregh}A - \text{antigh}A$$

Thus

$$\begin{aligned} \text{gh}z^a &= 0 \\ \text{gh}\eta^i &= 1 \\ \text{gh}\mathcal{P}_i &= -1 \end{aligned}$$

There is a canonical generator of the ghost number:

$$\mathcal{G} = i\eta^i \mathcal{P}_i$$

so that

$$\{A, G\} = i(\text{gh}A)A$$

We extend the resolution boundary operator δ to extended phase space by defining $\delta\eta^i = 0$. So δ has antighost number -1. We require that d has antighost number 0 and total ghost number 1. Then we take

$$d\mathcal{P}_i = \eta^j C_{ik}{}^k \mathcal{P}_k$$

which gives us

$$[\delta, d] = 0$$

This also gives us that d^2 is δ-exact.

Now the ghosts η^i are δ-closed but not δ-exact, so the homology of δ in $\mathbb{C}[\mathcal{P}_i] \otimes C^\infty(P) \otimes \mathbb{C}[\eta_i]$ is given by the original homology of δ tensored with the ghosts: $H^0(\delta) = C^\infty(\Sigma) \otimes \mathbb{C}[\eta^i]$, $H^i(\delta) = 0$ for $i > 0$. Thus δ gives a resolution of the exterior longitudinal algebra. Then the cohomology of d modulo δ is the cohomology of the longitudinal derivative restricted to Σ.

Now we finally come to BRST: the main theorem of homological perturbation theory tells us that in the situation we have constructed, there exists a differential $s = \delta + d +$ more of total ghost number 1 such that $s^2 = 0$ and $H^k(s) = H^k(d)$. In other words, we can find a differential on the resolution that replicates our desired cohomology. There is ambiguity in constructing s, but we can reduce the ambiguity and achieve something useful by requiring s to be a canonical transformation: we may find a fermionic function Ω such that

$$sx = \{x, \Omega\}$$

for any x in the BRST complex. Then s is unique up to a canonical change of variables. The function Ω satisfies $\{\Omega, \Omega\} = 0$ by the Jacobi identity.

With the introduction of BRST cohomology, we have solved our problem. Representatives of the cohomology classes allow us to work with our original phase space, and taking cohomology classes transports us to the physical part of the theory. The price we've had to pay is extending phase space to include ghosts and their momenta. But that's not so bad, is it?

You may have been wondering: how is all this related to equivariant cohomology? The answer is a bit messy, but from a bird's eye view, it makes sense. First, we'll have to realize that the cohomology of the longitudinal derivative was secretly a geometrical model of Lie algebra cohomology. In fact, mathematical treatments of BRST generally begin with an algebraic model for Lie algebra cohomology and skip the longitudinal geometry entirely. Here's the main point: the equivariant cohomology of a Lie algebra $\mathfrak{g}$ turns out to be the same as supersymmetrized Lie algebra cohomology of a corresponding graded Lie algebra $\mathfrak{g}_{\text{super}}$. The relation to the classical gauge theory case just described is a bit fuzzy to me still. In this case, the algebra of the ghosts and their momenta at a point furnish the exterior algebra of the Lie algebra, while the "super" part comes from promoting them to be differential forms on the constraint surface. This might not be exactly correct, but it's something along these lines.

4 BRST in Cohomological Field Theory

Now we'll move on to cohomological field theory, which is a type of topological field theory. These are field theories that are not manifestly topological, but in which we can identify a nilpotent operator Q such that physical observables are Q-cohomology classes and physical amplitudes are metric dependent because

Q-exact degrees of freedom decouple. One of the most famous cohomology field theories, and the one I'm most interested in, is the topologically twisted sigma model.

The paradigm for cohomological field theories is that their observables compute intersection theory of moduli spaces via physical methods. The moduli space takes the form

$$\mathcal{M} = \{\phi \in \mathcal{C} : D\phi = 0\} / G$$

where $\mathcal{C}$ is a space of fields, D is a differential operator, and G is a group of local transformations. Notice that this has a similar form to the physical phase space that we wanted to isolate in gauge theories. Because of this structure, a cohomological field theory is specified by fields, equations, and symmetries. The fields are a choice of chain complex. The equations single out interesting parts of the space of fields. The symmetries are typically an infinite dimensional Lie group that can be thought of as a gauge group. One example is Donaldson theory, where $\mathcal{C}$ is the space of connections on an SU(2)-vector bundle, D is the anti-self dual instanton equation, and G is the gauge group. The observables of Donaldson theory computes intersection theory on the moduli space of ASD instantons, and it turns out that these intersection numbers characterize differentiable structures on 4-manifolds. Other examples of moduli spaces include monopoles, metrics, and holomorphic maps. These spaces are generally finite dimensional, noncompact, and singular. To define a field theory, we need to compactify the moduli space and deal with the singularities. This is mathematically challenging and we won't talk about it today.

The connection to equivariant cohomology is fairly straightforward, unlike in the gauge theory case. The BRST operator in a topological field theory is the differential for a model of G-equivariant cohomology on a space of fields. In topological field theory, G is typically infinite dimensional.

Let's talk about the topological sigma A-model as an example, since that is one of the theories that inspired this talk. Let Σ be a surface with a metric h that induces a complex structure ϵ and X be an almost Kahler manifold with symplectic form ω and almost complex structure J. Then the space of fields of the A-model is $C^\infty(\Sigma, X)$, the equations are those for pseudo-holomorphic curves: $df + Jdf\epsilon = 0$, and the symmetries are trivial. Then we study differential forms on $C^\infty(\Sigma, X)$, and compute intersection numbers (whose integrals localize to the pseudo-holomorphic curves). The BRST cohomology in this case will restrict our integrals over the space of maps to integrals over the space of pseudo-holomorphic curves.

References

- Cordes, Moore, and Ramgoolam, *Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories*
- Henneaux and Teitolboim, *Quantization of Gauge Systems*
- Figueroa-O'Farrill, *BRST Cohomology*